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HOMOGENIZATION THEORY AND THE ASSESSMENT OF EXTREME FIELD VALUES IN COMPOSITES WITH RANDOM MICROSTRUCTURE*

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Abstract. Suitable macroscopic quantities are identified and used to assess the field distribution within a composite specimen of finite size with random microstructure. Composites made of N anisotropic dielectric materials are considered. The characteristic length scale of the microstructure relative to the length scale of the specimen is denoted by ε, and realizations of the random composite microstructure are labeled by ω. Consider any cube C_0 located inside the composite. The function $P^\varepsilon(t, C_0, \omega)$ gives the proportion of C_0 where the square of the electric field intensity exceeds t. The analysis focuses on the case when $0 < \varepsilon \ll 1$. Rigorous upper bounds on $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$ are found. They are given in terms of the macrofield modulation functions. The macrofield modulation functions capture the excursions of the local electric field fluctuations about the homogenized or macroscopic electric field. Information on the regularity of the macrofield modulations translates into bounds on $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$. Sufficient conditions are given in terms of the macrofield modulation functions that guarantee polynomial and exponential decay of $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$ with respect to "t." For random microstructure with oscillation on a sufficiently small scale we demonstrate that a pointwise bound on the macrofield modulation function provides a pointwise bound on the actual electric field intensity. These results are applied to assess the distribution of extreme electric field intensity for an L-shaped domain filled with a random laminar microstructure.

Key words. random composite materials, field fluctuations, material breakdown

AMS subject classifications. 35J15, 60G10, 78M40

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1. Introduction.

Failure of composite materials can often be attributed to the presence of large local fields. This includes extreme temperature gradients and large electric and current fields as well as mechanical stresses [9]. These fields are strongly influenced by the local microgeometry inside the composite. It is often the case that the microgeometry of heterogeneous specimens is known only in a statistical sense. Motivated by these considerations, we examine the distribution of extreme field values in random heterogeneous media. The focus here is to assess the likelihood that the magnitude of the electric field inside the composite exceeds a prescribed nominal value for almost every realization of the random microstructure.

Here we consider a random composite made up of N anisotropic dielectric materials with dielectric tensors $A_1, A_2, \ldots, A_N$. To describe the dielectric tensor for a finite size sample of random composite, we begin with the description of a random medium of infinite extent. The dielectric tensor field $A(\mathbf{y}, \omega)$ associated with the composite is a function of both position $\mathbf{y}$ and geometric realization ω taken from the sample space Ω. For each realization ω, the tensor field $A(\mathbf{y}, \omega)$ is piecewise constant taking

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only the values $A_1, A_2, \ldots, A_N$ for different points $\mathbf{y}$ in R^3. The random medium is assumed to be stationary, i.e., for any finite choice of points $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ and any vector $\mathbf{h}$, the distribution of the random tensor

$$A(\mathbf{y}_1 + \mathbf{h}, \omega), A(\mathbf{y}_2 + \mathbf{h}, \omega), \ldots, A(\mathbf{y}_k + \mathbf{h}, \omega) \tag{1.1}$$

does not depend on $\mathbf{h}$. The finite size composite specimen occupies the bounded domain $\mathcal{D}$, and points inside it are denoted by $\mathbf{x}$. The dielectric tensor for a composite with a random microstructure of characteristic length scale ε relative to the size of $\mathcal{D}$ is given by

$$A^\varepsilon(\mathbf{x}, \omega) = A\left(\frac{\mathbf{x}}{\varepsilon}, \omega\right). \tag{1.2}$$

The potential inside the composite is denoted by $\phi^\varepsilon(\mathbf{x}, \omega)$. For a prescribed charge distribution $f = f(\mathbf{x})$ and prescribed values of the electric potential on the boundary of the domain $\mathcal{D}$ given by $\phi^\varepsilon(\mathbf{x}, \omega) = \phi_0(\mathbf{x})$, the potential is the solution of

$$-\mathrm{div}(A^\varepsilon(\mathbf{x}, \omega)\nabla\phi^\varepsilon(\mathbf{x}, \omega)) = f \tag{1.3}$$

in $\mathcal{D}$. Here (1.3) holds in the sense of distributions. The associated electric field $\mathbf{E}^\varepsilon(\mathbf{x}, \omega) = -\nabla\phi^\varepsilon(\mathbf{x}, \omega)$ is not necessarily a stationary random field; this is due to the finite size of the domain $\mathcal{D}$ and the prescribed charge distribution.

Failure initiation criteria are often given in terms of a critical field strength such that if a significant portion of the sample has field strength above this value, then the failure process is initiated [7]. Motivated by this observation, we focus on the subset of the composite where $|\mathbf{E}^\varepsilon|^2$ exceeds the value $t > 0$, and we denote it by $S_t^\varepsilon(\omega)$. Consider any cube C_0 inside the composite. It is assumed here that the boundary of the cube does not intersect the boundary of the specimen. The field distribution function $\lambda^\varepsilon(t, C_0, \omega)$ gives the volume of the intersection of $S_t^\varepsilon(\omega)$ with C_0, i.e., $\lambda^\varepsilon(t, C_0, \omega) = |S_t^\varepsilon(\omega) \cap C_0|$. Here $|S|$ denotes the volume of the set S. Division of $\lambda^\varepsilon(t, C_0, \omega)$ by the volume of the cube gives the function $P^\varepsilon(t, C_0, \omega)$. Here $P^\varepsilon(t, C_0, \omega)$ gives the proportion of the cube experiencing field strength greater than t. One also defines the electric field distribution inside the part of the ith phase contained in the cube C_0. The volume of the set in the ith phase contained in C_0 where $|\mathbf{E}^\varepsilon|^2$ exceeds the value $t > 0$ is denoted by $\lambda_i^\varepsilon(t, C_0, \omega)$. The set occupied by the ith phase is denoted by $S_i^\varepsilon(\omega)$. Analogously $P_i^\varepsilon(t, C_0, \omega) \equiv \lambda_i^\varepsilon(t, C_0, \omega)/|S_i^\varepsilon(\omega) \cap C_0|$ gives the proportion if the ith phase contained in C_0 with field strength greater than t.

In this paper we obtain bounds on $P^\varepsilon(t, C_0, \omega)$ and $P_i^\varepsilon(t, C_0, \omega)$ in the limit of vanishing ε. These bounds are expressed in terms of suitable macroscopic quantities dubbed macrofield modulation functions. To illustrate the ideas, one applies the Chebyshev inequality to obtain the bound on $P^\varepsilon(t, C_0, \omega)$ given by

$$P^\varepsilon(t, C_0, \omega) \leq t^{-p}\frac{1}{|C_0|}\int_{C_0} |\mathbf{E}^\varepsilon(\mathbf{x}, \omega)|^{2p}\, d\mathbf{x}. \tag{1.4}$$

In section 2 we state the homogenized version of (1.4) given by

$$\lim_{\varepsilon \to 0} P^\varepsilon(t, C_0, \omega) \leq t^{-p} A_p(C_0). \tag{1.5}$$

Here $A_p(C_0)$ is independent of ω and is described in terms of the macrofield modulation functions. The macrofield modulation of order p is the L^p norm of the square of the electric field intensity for the associated corrector problem (2.2) posed on the infinite random medium when the random medium is subjected to an imposed macroscopic electric field; see (2.9). Proposition 2.1 explicitly shows how integrability of order p at the level of the corrector problem contributes to the t^{-p} order decay of $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$. Similarly, Proposition 2.3 shows how L^∞ regularity of the square of the electric field intensity for the associated corrector problem allows $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$ to vanish above a critical value of t. For this case we can pass to a subsequence, if necessary, to derive a pointwise bound on the local electric field intensity for almost every realization of the random microstructure when the scale of the microstructure is sufficiently small; see Proposition 2.4. When the macrofield modulation function has bounded mean oscillation, an explicit upper bound is obtained that is exponential in $-t$ and is given in terms of the BMO norm of the macrofield modulation function; see Proposition 2.5. The corrector problem that is used to define the macrofield modulation functions is well known and naturally arises in the definition of the effective dielectric tensor [1, 10, 17, 18].

It is pointed out that the main results given by Propositions 2.1 through 2.6 are strong limit theorems in that they hold for almost all realizations of the random medium. Propositions 2.1 through 2.6 are a direct consequence of the homogenization constraints given in Proposition 3.1. These constraints relate the macrofield modulation functions to the distribution of states for the square of the electric field intensity. This type of constraint is introduced in [11, 14] for the case of graded locally periodic microstructures and in the context of G convergence for multiphase linearly elastic composites. The results reported here apply to the mathematically identical situations appearing in the contexts of thermal conductivity and DC electric conductivity.

The paper is organized as follows: In section 2 the macrofield modulation functions are introduced and the main results are presented. The homogenization constraint is introduced and derived in section 3. The homogenized version of Chebyshev's inequality is established in section 4. The bounds on the support of $\lim_{\varepsilon\to 0} P_i^\varepsilon(t, C_0, \omega)$ and $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$ are obtained in section 5. These are given in terms of the L^∞ norm of the macrofield modulation functions. The pointwise upper bounds are derived in section 6. The exponentially decaying bound on $\lim_{\varepsilon\to 0} P^\varepsilon(t, C_0, \omega)$ is derived in section 7. In section 8 we consider a highly oscillatory, randomly layered dielectric occupying an L-shaped domain. The dielectric is subjected to a prescribed charge density and the electric potential satisfies homogeneous Dirichlet boundary conditions. The macrofield modulation functions together with the results of section 2 are applied to assess the distribution of the electric field intensity inside the domain.

2. The macrofield modulation functions and main results.

To introduce the macrofield modulation functions, we consider a random composite of infinite extent. For stationary random media it is shown in [17] that one can regard the dielectric tensor $A(\mathbf{y}, \omega)$ as the realization of a random function $\tilde{A}$ with respect to a three-dimensional dynamical system T acting on a suitable sample space; see also [2] for a more recent discussion. In view of this let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space. For a given partition of Ω into N measurable subsets $\Omega_1, \Omega_2, \dots, \Omega_N$ we introduce the indicator functions $\tilde{\chi}_i$ taking the values 1 in Ω_i and zero outside and set $\tilde{A}(\omega) = \sum_{i=1}^N A_i \tilde{\chi}_i(\omega)$. Following [5, 10, 17] we regard the dielectric $A(\mathbf{y}, \omega)$ as a realization of $\tilde{A}$ with respect to a three-dimensional dynamical system T on Ω, i.e., $A(\mathbf{y}, \omega) = \tilde{A}(T(\mathbf{y})\omega)$ for $(\mathbf{y}, \omega)$ in $R^3 \times \Omega$. Here the family of mappings $T = T(\mathbf{y})$, $\mathbf{y}$ in R^3 from Ω into Ω, is one to one and preserves the measure $\mathcal{P}$ on Ω; i.e., for any A in $\mathcal{F}$ one has $\mathcal{P}(T(-\mathbf{y})A) = \mathcal{P}(A)$. The family of transforms is a group with $T(0)\omega = \omega$, $T(\mathbf{y} + \mathbf{h}) = T(\mathbf{y})T(\mathbf{h})$, and for any $\mathcal{P}$ measurable function $\tilde{f}$ on Ω, the function $\tilde{f}(T(\mathbf{y})\omega)$ defined on $R^3 \times \Omega$ is also measurable with respect to $\mathcal{L} \times \mathcal{F}$, where $\mathcal{L}$ stands for the σ-algebra of Lebesgue-measurable subsets of R^3. Lastly, it is assumed that the dynamical system is ergodic.

Let $\mathbf{e}^1$, $\mathbf{e}^2$, $\mathbf{e}^3$ represent unit vectors along the coordinate directions in R^3. A constant electric field $\mathbf{e}^k$ is imposed on the infinite random medium. The dielectric response in the composite is given by an electric field that can be decomposed into the imposed electric field $\mathbf{e}^k$ and a stationary random fluctuation $-\nabla\varphi^k(\mathbf{y},\omega) = \mathbf{G}^k(T(\mathbf{y})\omega)$, where $\mathbf{G}^k$ is in $L^2(\Omega,\mathcal{P})$ with zero mean, i.e., $\langle \mathbf{G}^k \rangle = \int_\Omega \mathbf{G}^k d\mathcal{P} = 0$; see [5, 10, 17, 8]. From the Birkhoff ergodic theorem it follows that for any sequence of cubes $Q(r)$ of side length $2r$ and volume $|Q(r)|$,

$$\lim_{r\to\infty} \frac{1}{|Q(r)|} \int_{Q(r)} (-\nabla\varphi^k(\mathbf{y},\omega))\, d\mathbf{y} = \langle \mathbf{G}^k \rangle = 0. \tag{2.1}$$

The fluctuation solves

$$-\mathrm{div}(A(\mathbf{y},\omega)(\nabla\varphi^k(\mathbf{y},\omega) + \mathbf{e}^k)) = 0 \tag{2.2}$$

for $\mathbf{y}$ in R^3. For an imposed constant electric field of the general form $\overline{\mathbf{E}} = (E_1\mathbf{e}^1 + E_2\mathbf{e}^2 + E_3\mathbf{e}^3)$, the stationary random fluctuation is obtained by superposition and is given by $-\nabla\varphi(\mathbf{y},\omega) = \sum_{k=1}^3 E_k \mathbf{G}^k(T(\mathbf{y})\omega)$. For future reference we introduce the matrix with column vectors $\mathbf{G}^k$ given by $\underline{\tilde{\mathbf{G}}}(\omega) = (\mathbf{G}^1(\omega), \mathbf{G}^2(\omega), \mathbf{G}^3(\omega))$. Then $-\nabla\varphi(\mathbf{y},\omega) = \underline{\tilde{\mathbf{G}}}(T(\mathbf{y})\omega)\overline{\mathbf{E}}$ and $\mathbf{E}(\mathbf{y},\omega) = (I + \underline{\tilde{\mathbf{G}}}(T(\mathbf{y})\omega))\overline{\mathbf{E}}$. The dielectric displacement is a stationary random field, and its mean is given by

$$\langle \mathbf{D} \rangle = \int_\Omega \tilde{A}(\omega)(I + \underline{\tilde{\mathbf{G}}}(\omega))\overline{\mathbf{E}}\, d\mathcal{P}(\omega) = \lim_{r\to\infty} \frac{1}{|Q(r)|} \int_{Q(r)} A(\mathbf{y},\omega)\mathbf{E}(\mathbf{y},\omega)\, dy. \tag{2.3}$$

The effective dielectric tensor A^E provides the linear relation between the imposed electric field $\overline{\mathbf{E}}$ and the mean dielectric displacement $\langle \mathbf{D} \rangle$, i.e., $\langle \mathbf{D} \rangle = A^E\overline{\mathbf{E}}$; see [5, 10, 17, 8].

When considering failure initiation it is important to assess the magnitude of the local electric field inside the random medium arising from the imposed electric field $\overline{\mathbf{E}}$. Here one is interested in the probability that the square of the electric field intensity $|\mathbf{E}|^2$ in the ith phase exceeds a nominal value t. For the stationary random case this probability is the same for every point and is given by $\theta_{t,i} = \mathcal{P}(\tilde{\chi}_i(\omega)|(I + \underline{\tilde{\mathbf{G}}}(\omega))\overline{\mathbf{E}}|^2 > t)$. Other quantities that are useful for local field assessment are given by the L^p norms, $1 \le p \le \infty$. The $L^p(\Omega)$ norm of a $\mathcal{P}$ measurable function $\tilde{g}$ is denoted by $\|\tilde{g}\|_{L^p(\Omega)}$. Since $T(\mathbf{y})$ preserves the measure $\mathcal{P}$ on Ω, it follows that

$$\|\chi_i(\mathbf{y},\omega)|\mathbf{E}(\mathbf{y},\omega)|^2\|_{L^p(\Omega)} = \|\tilde{\chi}_i(\omega)|(I + \underline{\tilde{\mathbf{G}}}(\omega))\overline{\mathbf{E}}|^2\|_{L^p(\Omega)} \quad \text{for every } \mathbf{y} \text{ in } R^3. \tag{2.4}$$

Motivated by these considerations, we introduce moments of the local electric field of order p.

Definition: Moments of the local electric field.

$$f_p^i(\overline{\mathbf{E}}) = \|\tilde{\chi}_i(\omega)|(I + \underline{\tilde{\mathbf{G}}}(\omega))\overline{\mathbf{E}}|^2\|_{L^p(\Omega)} = \left(\int_\Omega \tilde{\chi}_i(\omega)|(I + \underline{\tilde{\mathbf{G}}}(\omega))\overline{\mathbf{E}}|^{2p}\, d\mathcal{P}(\omega)\right)^{1/p} \tag{2.5}$$

for $1 \le p \le \infty$.

Moments of the electric field have been calculated for two-dimensional random dispersions of disk-, needle-, and square-shaped inclusions in [4].

It is pointed out that the electric field generated by a constant imposed electric field is self-similar under a rescaling of the infinite random medium. Indeed, set $\varepsilon_k = 1/k$ and rescale the material properties by $A^{\varepsilon_k}(\mathbf{y},\omega) = A(\mathbf{y}/\varepsilon_k,\omega)$. It is easily checked that the electric field also rescales as $\mathbf{E}^{\varepsilon_k}(\mathbf{y},\omega) = \mathbf{E}(\mathbf{y}/\varepsilon_k,\omega)$. Thus the analysis of electric field distribution for the ε_k scale microstructure reduces to an analysis for the unrescaled random media. However, this symmetry is broken for generic situations when the specimen is finite in extent and the loading is not uniform throughout the sample. Because of this the electric field in the composite is not obtained directly through an analysis of the electric field in an infinite random medium. Instead, it is shown here that a suitable multiscale analysis using macrofield modulation functions provides rigorous bounds on the field distributions $P^{\varepsilon}(t, C_0, \omega)$ and $P_i^{\varepsilon}(t, C_0, \omega)$ for almost every realization in the limit of vanishing ε.

Consider a finite size specimen $\mathcal{D}$ filled with random composite with characteristic length scale $\varepsilon_k = 1/k$. Here the composite is described by $A^{\varepsilon_k}(\mathbf{x},\omega) = \tilde{A}(T(\mathbf{x}/\varepsilon_k)\omega)$, and the electric potential $\phi^{\varepsilon_k}(\mathbf{x},\omega)$ solves the boundary value problem described in the introduction with equilibrium condition given by (1.3). The electric field is given by $\mathbf{E}^{\varepsilon_k} = -\nabla(\phi^{\varepsilon_k})$. The multiscale analysis proceeds in two steps. The first step is the up scaling or homogenization step where the macroscopic electric field is determined. From the theory of random homogenization, the fields $\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)$ and $\mathbf{D}^{\varepsilon_k}(\mathbf{x},\omega) = \mathbf{A}^{\varepsilon_k}(\mathbf{x},\omega)\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)$ converge to the deterministic macroscopic fields $\mathbf{E}(\mathbf{x})^M$ and $\mathbf{D}^M(x)$ as ε_k goes to zero for almost every ω; see [10, 17]. Here the convergence of the sequences of electric and displacement fields is given by weak convergence in $L^2(\mathcal{D})^3$. The deterministic macroscopic potential $\phi^M(\mathbf{x})$ satisfies the boundary condition $\phi^M(\mathbf{x}) = \phi_0(\mathbf{x})$. The macroscopic dielectric displacement satisfies the equilibrium equation

$$\text{div}\mathbf{D}^M = f \tag{2.6}$$

and $\mathbf{E}^M = -\nabla\phi^M$. The displacement and electric field are related through the homogenized constitutive law

$$\mathbf{D}^M(\mathbf{x}) = A^E\mathbf{E}^M(\mathbf{x}). \tag{2.7}$$

The second step is a down scaling step and gives the interaction between the macroscopic electric field $\mathbf{E}^M(\mathbf{x})$ and the microstructure. For each $\mathbf{x}$, the microscopic dielectric response is given by

$$\mathbf{E}(\mathbf{x},\mathbf{y},\omega) = (I + \underline{\tilde{\mathbf{G}}}(T(\mathbf{y})\omega))\mathbf{E}^M(\mathbf{x}). \tag{2.8}$$

The relevant interaction is described by the macrofield modulation function $f_p^i(\mathbf{E}^M(\mathbf{x}))$ given by the following definition.

Definition: Macrofield modulation function.

$$f_p^i(\mathbf{E}^M(\mathbf{x})) = \|\tilde{\chi}_i(\omega)|(I + \underline{\tilde{\mathbf{G}}}(\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^p(\Omega)} \tag{2.9}$$

for $1 \le p \le \infty$. The macrofield modulation function $f_p^i(\mathbf{E}^M(\mathbf{x}))$ provides a measure of the amplification or diminution of $\mathbf{E}^M(\mathbf{x})$ by the random medium. Explicit formulas for the macrofield modulation functions for randomly layered two-phase dielectrics are given in section 8.

Consider any cube C_0 inside the composite. The L^1 norm of a function $g(\mathbf{x})$ over the cube C_0 is denoted by $\|g\|_{L^1(C_0)}$. In what follows, it is always assumed that

$\theta_i = \int_\Omega \tilde{\chi}_i(\omega)\, d\mathcal{P} > 0$, and from ergodicity the volume occupied by the ith phase in the cube C_0 tends to the nonzero limit $\lim_{\varepsilon_k \to 0} \int_{C_0} \tilde{\chi}_i(T(\mathbf{x}/\varepsilon_k)\omega)\, d\mathbf{x} = \theta_i |C_0|$ as ε_k tends to zero. Passing to a subsequence, if necessary, we consider $\lim_{\varepsilon_k \to 0} P_i^{\varepsilon_k}(t, C_0, \omega)$.

If it is known that $\| |f_p^i(\mathbf{E}^M(\mathbf{x}))|^p \|_{L^1(C_0)} < \infty$ for some p, then the following proposition shows that $\lim_{\varepsilon_k \to 0} P_i^{\varepsilon_k}(t, C_0, \omega)$ decays on the order of t^{-p}.

Proposition 2.1 (homogenization of Chebyshev's inequality). *Given that*

$$\| |f_p^i(\mathbf{E}^M(\mathbf{x}))|^p \|_{L^1(C_0)} < \infty$$

for some p with $1 \le p < \infty$, then for almost every realization ω one has

$$\begin{aligned} \lim_{\varepsilon_k \to 0} P_i^{\varepsilon_k}(t, C_0, \omega) &\le t^{-p} \frac{1}{\theta_i |C_0|} \| |f_p^i(\mathbf{E}^M(\mathbf{x}))|^p \|_{L^1(C_0)} \\ &= t^{-p} \frac{1}{\theta_i |C_0|} \int_{C_0} \int_\Omega \tilde{\chi}_i(\omega) |(I + \underline{\tilde{\mathbf{G}}}(\omega)) \mathbf{E}^M(\mathbf{x})|^{2p}\, d\mathcal{P}(\omega)\, d\mathbf{x}. \end{aligned} \tag{2.10}$$

If $\| |f_p^i(\mathbf{E}^M(\mathbf{x}))|^p \|_{L^1(C_0)} < \infty$ for all $i = 1, 2, \ldots, N$, then

$$\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \le t^{-p} \frac{1}{|C_0|} \int_{C_0} \int_\Omega |(I + \underline{\tilde{\mathbf{G}}}(\omega)) \mathbf{E}^M(\mathbf{x})|^{2p}\, d\mathcal{P}(\omega)\, d\mathbf{x} \tag{2.11}$$

for almost every realization ω.

It is clear that the coefficients of t^{-p} in (2.10) and (2.11) depend upon the Dirichlet data ϕ_0, charge density f, and the domain $\mathcal{D}$ through the solution of the homogenized problem (2.6). The proof of Proposition 2.1 is given in section 4.

The L^∞ norm of a function $g(\mathbf{x})$ over the cube C_0 is denoted by $\|g\|_{L^\infty(C_0)}$. A characterization of the set of parameters t where $\lim_{\varepsilon_k \to 0} P_i^{\varepsilon_k}(t, C_0, \omega)$ vanishes for almost every realization is given in the following proposition.

Proposition 2.2. *If $t > \|f_\infty^i(\mathbf{E}^M(\mathbf{x}))\|_{L^\infty(C_0)}$, then $\lim_{\varepsilon_k \to 0} P_i^{\varepsilon_k}(t, C_0, \omega) = 0$ for almost every ω in Ω*

From the proposition it is evident that if $t > \|f_\infty^i(\mathbf{E}^M(\mathbf{x}))\|_{L^\infty(C_0)}$, then the volume of the subsets in the ith phase for which $|\mathbf{E}^{\varepsilon_k}(\mathbf{x}, \omega)|^2 > t$ vanishes as ε_k tends to zero with probability one. The proof of Proposition 2.2 is given in section 5.

We introduce the macrostress modulation $M(\mathbf{E}^M(\mathbf{x}))$ given by

$$M(\mathbf{E}^M(\mathbf{x})) = \max_{i=1,\ldots,N} f_\infty^i(\mathbf{E}^M(\mathbf{x})) \tag{2.12}$$

and characterize $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega)$ in a way analogous to Proposition 2.2. This is stated in the following proposition.

Proposition 2.3. *If $t > \|M(\mathbf{E}^M(\mathbf{x}))\|_{L^\infty(C_0)}$, then $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) = 0$ for almost every realization.*

For random microstructure with oscillation on a sufficiently small scale, it is found that a pointwise bound on the macrofield modulation function delivers a pointwise bound on the actual electric field intensity for almost every realization of the microstructure.

Proposition 2.4 (pointwise bounds on the electric field intensity). *Suppose that*

$$t > M(\mathbf{E}^M(\mathbf{x})) \tag{2.13}$$

on C_0. Then one can pass to a subsequence $\{\varepsilon_{k'}\}_{k'=1}^\infty$ if necessary to find that there is a critical ε_0 such that for every $\varepsilon_{k'} < \varepsilon_0$,

$$|\mathbf{E}^{\varepsilon_{k'}}(\mathbf{x}, \omega)|^2 \le t \tag{2.14}$$

for almost every $\mathbf{x}$ *in* C_0 *and for almost every realization* ω. *Here* ε_0 *can depend upon* $\mathbf{x}$ *and* ω.

The proof of Proposition 2.4 is given in section 6.

Last, we give conditions for which $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega)$ decreases exponentially with t. To do this we introduce the BMO norm of $M(\mathbf{E}^M(\mathbf{x}))$ over the cube C_0 given by

$$\|M\|_{BMO} = \sup_{C \subset C_0} \left(\frac{1}{|C|} \int_C |M(\mathbf{E}^M(\mathbf{x})) - M_C| \, d\mathbf{x} \right), \tag{2.15}$$

where M_C is the average of $M(\mathbf{E}^M(\mathbf{x}))$ over C and the supremum is taken over all subcubes C of C_0. The BMO norm and the space of functions of bounded mean oscillation were introduced by John and Nirenberg [6]. The space of functions with bounded L^∞ norm are a subspace of the functions with bounded BMO norm since $\|M(\mathbf{E}^M)\|_{BMO} \leq c\|M(\mathbf{E}^M)\|_{L^\infty(C_0)}$, where c is a constant depending on C_0.

For any positive number α between zero and one, we define the constant $C(\alpha)$ by

$$C(\alpha) = \frac{\alpha |\ln \alpha|}{8\|M\|_{BMO}}. \tag{2.16}$$

With the average of $M(\mathbf{E}^M(\mathbf{x}))$ over the cube C_0 denoted by M_{C_0}, the bound on $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega)$ is given in the following proposition.

Proposition 2.5. *If* $t > 8\|M\|_{BMO}\alpha^{-1} + M_{C_0}$, *then*

$$\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \leq \alpha^{-1} e^{-C(\alpha) \times (t - M_{C_0})} \tag{2.17}$$

for almost every realization.

For t fixed the proposition shows that $P^{\varepsilon_k}(t, C_0, \omega)$ approaches or drops below

$$\alpha^{-1} e^{-C(\alpha) \times (t - M_{C_0})}$$

for ε_k sufficiently small for almost every realization. It also shows that the upper bound is exponentially decreasing for large t. Optimization over α (see section 7) provides the tighter upper bound given by the following proposition.

Proposition 2.6. *If* $t > 8\|M\|_{BMO} + M_{C_0}$, *then for almost every realization of the random medium*

$$\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \leq (\alpha(t))^{-1} e \times e^{[-\alpha(t)(t - M_{C_0})/(8\|M\|_{BMO})]}, \tag{2.18}$$

where the factor $\alpha(t)$ *lies in the interval* $e^{-1} < \alpha(t) < 1$ *and is the root of the equation*

$$\kappa^{-1} - \alpha(1 + \ln \alpha) = 0, \tag{2.19}$$

with $\kappa = (t - M_{C_0})/(8\|M\|_{BMO})$.

It is pointed out that if the macroscopic electric field $\mathbf{E}^M$ is constant inside C_0, then $\|M\|_{BMO} = 0$, $M_{C_0} = M(\mathbf{E}^M) = \|M(\mathbf{E}^M)\|_{L^\infty(C_0)}$, and Propositions 2.3, 2.5, and 2.6 reduce to the observation that if $t > M(\mathbf{E}^M)$, then $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) = 0$ for almost all ω.

Propositions 2.1 through 2.6 provide the opportunity to recover information on the behavior of the electric field intensity $|\mathbf{E}^\varepsilon(\mathbf{x}, \omega)|$ inside the random microstructure from knowledge of the behavior of the macrofield modulation functions. An application is given in section 8 where the electric field distribution inside an L-shaped domain containing a highly oscillatory random laminate is analyzed.

3. Homogenization constraints. The homogenization constraints are motivated by considering the case of a random composite of infinite extent. For the $p=\infty$ case the homogenization constraint follows immediately from the definition of $f^i_\infty(\overline{\mathbf{E}})$. Indeed, it is clear from the definition of the L^∞ norm that $t \geq f^i_\infty(\overline{\mathbf{E}})$ implies that $\theta_{t,i}=0$, and equivalently, if $\theta_{t,i}>0$, it follows that $f^i_\infty(\overline{\mathbf{E}})>t$. This delivers the homogenization constraints given by

$$\theta_{t,i}(f^i_\infty(\overline{\mathbf{E}})-t) \geq 0. \tag{3.1}$$

For $1 \leq p < \infty$, Chebyshev's inequality implies

$$t^{-p}(f^i_p(\overline{\mathbf{E}}))^p \geq \theta_{t,i}. \tag{3.2}$$

Inequalities (3.1) and (3.2) are the specialization of the homogenization constraints to stationary random composites of infinite extent. In the general context the macroscopic electric field is not uniform and the composite specimen has finite size. For general specimen shapes and nonuniform loading, the constraints analogous to (3.1) and (3.2) are given in terms of $f^i_\infty(\mathbf{E}^M(\mathbf{x}))$ and $f^i_p(\mathbf{E}^M(\mathbf{x}))$. In order to complete the description of the homogenization constraint, a suitable generalization of $\theta_{t,i}$ is needed. For this case, one considers a realization of the random composite $A^{\varepsilon_k}(\mathbf{x},\omega)$ and the set in the ith phase where the square of the electric field intensity $|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2$ exceeds t is denoted by $S^{\varepsilon_k}_{t,i}(\omega)$. Consider any subdomain Q of the specimen such that the boundary of Q does not intersect the boundary of the specimen. The distribution function $\lambda^{\varepsilon_k}_i(t,Q,\omega)$ is defined by $\lambda^{\varepsilon_k}_i(t,Q,\omega)=|S^{\varepsilon_k}_{t,i}(\omega)\cap Q|$. The indicator function for the set $S^{\varepsilon_k}_{t,i}(\omega)$ is written $\chi^{\varepsilon_k}_{t,i}(\mathbf{x},\omega)$ taking the value 1 in $S^{\varepsilon_k}_{t,i}(\omega)$ and 0 outside and we write $\lambda^{\varepsilon_k}_i(t,Q,\omega)=\int_Q \chi^{\varepsilon_k}_{t,i} d\mathbf{x}$. From the theory of weak convergence there exists a (Lebesgue measurable) density $\theta_{t,i}(\mathbf{x},\omega)$ taking values in the interval $[0,1]$ such that (on passage to a subsequence if necessary) $\lim_{k\to\infty}\lambda^{\varepsilon_k}_i(t,Q,\omega)=\int_Q \theta_{t,i}(\mathbf{x},\omega)dx$. The density $\theta_{t,i}(\mathbf{x},\omega)$ is the local distribution of states of the square of the electric field intensity $|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2$ in the ith phase as ε_k goes to zero. Here, the random fields $\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)$ and $\theta_{t,i}(\mathbf{x},\omega)$ can no longer be regarded as stationary; this is due to the finite size of the domain and nonuniform charge distribution within the dielectric. However, for almost every realization one has the homogenization constraints given in the following proposition.

Proposition 3.1 (homogenization constraints). *For almost every point $\mathbf{x}$ in Q and almost every realization ω in Ω, one has*

$$\theta_{t,i}(\mathbf{x},\omega)(f^i_\infty(\mathbf{E}^M(\mathbf{x}))-t) \geq 0, \quad i=1,\dots,N, \tag{3.3}$$

and for $1/q+1/p=1$,

$$\theta^{1/q}_{t,i}(\mathbf{x},\omega)f^i_p(\mathbf{E}^M(\mathbf{x})) \geq t\theta_{t,i}(\mathbf{x},\omega), \quad i=1,\dots,N. \tag{3.4}$$

It is clear that (3.3) and (3.4) are the extensions of (3.1) and (3.2) to situations where the macroscopic electric field is no longer uniform.

Proof. For a given realization ω, it follows from the definition of the set $S^{\varepsilon_k}_{t,i}(\omega)$ that

$$\chi^{\varepsilon_k}_{t,i}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2 - t\chi^{\varepsilon_k}_{t,i}(\mathbf{x},\omega) > 0. \tag{3.5}$$

Multiplying (3.5) by any nonnegative test function $p(\mathbf{x})$ and integrating over $\mathcal{D}$ gives

$$\int_{\mathcal{D}} p(\mathbf{x})(\chi^{\varepsilon_k}_{t,i}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2 - t\chi^{\varepsilon_k}_{t,i}(\mathbf{x},\omega))\, d\mathbf{x} > 0. \tag{3.6}$$

Taking limits and passing to subsequences if necessary gives

$$\lim_{\varepsilon_k\to 0}\int_{\mathcal{D}} p(\mathbf{x})\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2\,d\mathbf{x} \geq t\int_{\mathcal{D}} p(\mathbf{x})\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x}. \tag{3.7}$$

We will use the following lemma.

LEMMA 3.2.

$$\int_{\mathcal{D}} p(\mathbf{x})f_\infty^i(\mathbf{E}^M(\mathbf{x}))\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x} \geq \lim_{\varepsilon_k\to 0}\int_{\mathcal{D}} p(\mathbf{x})\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2\,d\mathbf{x}, \tag{3.8}$$

and for $1/q + 1/p = 1$*,*

$$\int_{\mathcal{D}} p(\mathbf{x})f_p^i(\mathbf{E}^M(\mathbf{x}))\theta_{t,i}^{1/q}(\mathbf{x},\omega)\,d\mathbf{x} \geq \lim_{\varepsilon_k\to 0}\int_{\mathcal{D}} p(\mathbf{x})\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2\,d\mathbf{x} \tag{3.9}$$

for all nonnegative $p(\mathbf{x})$ *in* $C_0^\infty(\mathcal{D})$ *and for almost every* ω*.*

Applying the inequality (3.7) together with Lemma 3.2 delivers

$$\int_{\mathcal{D}} p(\mathbf{x})f_\infty^i(\mathbf{E}^M(\mathbf{x}))\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x} \geq t\int_{\mathcal{D}} p(\mathbf{x})\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x} \tag{3.10}$$

and

$$\int_{\mathcal{D}} p(\mathbf{x})f_p^i(\mathbf{E}^M(\mathbf{x}))\theta_{t,i}^{1/q}(\mathbf{x},\omega)\,d\mathbf{x} \geq t\int_{\mathcal{D}} p(\mathbf{x})\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x} \tag{3.11}$$

for almost every ω. The proposition now follows since (3.10) and (3.11) hold for every nonnegative test function. ∎

Proof of Lemma 3.2. We write

$$A^{\varepsilon_k}(\mathbf{x},\omega) = A^{\varepsilon_k}(A_1, A_2, \ldots, A_N, \mathbf{x}, \omega) = \sum_{\ell=1}^{N} \tilde{\chi}_\ell(T(\mathbf{x}/\varepsilon_k)\omega)A_\ell. \tag{3.12}$$

We introduce the $N+1$ phase composite identical to the previous except that in $S_{t,i}^{\varepsilon_k}(\omega)$ it has dielectric constant P_{N+1}. The piecewise constant dielectric tensor for this composite is given by

$$\begin{aligned}\hat{A}^{\varepsilon_k}(\mathbf{x},\omega) &= \hat{A}^{\varepsilon_k}(A_1, A_2, \ldots, A_N, P_{N+1}, \mathbf{x}, \omega)\\ &= \sum_{\substack{\ell=1\\ \ell\neq i}}^{N} \tilde{\chi}_\ell(T(\mathbf{x}/\varepsilon_k)\omega)A_\ell\\ &\quad + \tilde{\chi}_i(T(\mathbf{x}/\varepsilon_k)\omega)(1-\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega))A_i + \tilde{\chi}_i(T(\mathbf{x}/\varepsilon_k)\omega)\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)P_{N+1}.\end{aligned} \tag{3.13}$$

For P_{N+1} in a neighborhood of A_i, we invoke the compactness property of G-convergence with respect to the sequence $\{\hat{A}^{\varepsilon_k}(A_1, A_2, \ldots, A_N, P_{N+1}, \mathbf{x}, \omega)\}_{k=1}^\infty$ [19, 16] to assert the existence of a G-converging subsequence also denoted by

$$\{\hat{A}^{\varepsilon_k}(A_1, A_2, \ldots, A_N, P_{N+1}, \mathbf{x}, \omega)\}_{k=1}^\infty$$

and a G-limit denoted by $\hat{A}^E(A_1, A_2, \ldots, A_N, P_{N+1}, \mathbf{x}, \omega)$. The partial derivatives of $\hat{A}^E(A_1, A_2, \ldots, A_N, P_{N+1}, \mathbf{x}, \omega)$ with respect to each element of P_{N+1} evaluated at $P_{N+1} = A_i$ are given by [11, 12, 13]:

$$\begin{aligned}&\nabla_{mn}^{N+1}\hat{A}_{op}^E(A_1, A_2, \ldots, A_N, A_i, \mathbf{x}, \omega)\\ &\quad = \lim_{r\to 0}\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r)|}\int_{Q(\mathbf{x},r)} \chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)(\partial_m w_o^{k,r} + \mathbf{e}_m^o)(\partial_n w_p^{k,r} + \mathbf{e}_n^p)\,d\mathbf{y}\right).\end{aligned} \tag{3.14}$$

Here $Q(\mathbf{x},r)$ is a cube of side length $2r$ inside $\mathcal{D}$ centered at $\mathbf{x}$ with volume given by $|Q(\mathbf{x},r)|$, and the functions $w_p^{k,r}$ vanish on the boundary of the cube and are the solutions of

$$-\text{div}(A^{\varepsilon_k}(\mathbf{y},\omega)(\nabla w_p^{k,r}(\mathbf{y})+\mathbf{e}^p))=0, \quad p=1,2,3, \tag{3.15}$$

for $\mathbf{y}$ in $Q(\mathbf{x},r)$. From [11, 12, 13] one has for every test function p vanishing on the boundary of $\mathcal{D}$ that

$$\begin{aligned}&\lim_{\varepsilon_k\to 0}\int_{\mathcal{D}} p(\mathbf{x})\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2\,d\mathbf{x}\\ &\quad=\int_{\mathcal{D}} p(\mathbf{x})\left(\sum_{m=1}^{3}\nabla_{mm}^{N+1}\hat{A}^E(A_1,A_2,\dots,A_N,A_i,\mathbf{x},\omega)\right)\mathbf{E}^M(\mathbf{x})\cdot\mathbf{E}^M(\mathbf{x})\,d\mathbf{x}.\end{aligned}\tag{3.16}$$

Here

$$\begin{aligned}&\left(\sum_{m=1}^{3}\nabla_{mm}^{N+1}\hat{A}^E(A_1,A_2,\dots,A_N,A_i,\mathbf{x},\omega)\right)\mathbf{E}^M(\mathbf{x})\cdot\mathbf{E}^M(\mathbf{x})\\ &\quad=\sum_{op}\left(\lim_{r\to 0}\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r)|}\int_{Q(\mathbf{x},r)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)(\nabla w_o^{k,r}+\mathbf{e}^o)\right.\right.\\ &\qquad\left.\left.\cdot(\nabla w_p^{k,r}+\mathbf{e}^p)\,d\mathbf{y}\right)\mathbf{E}_o^M(\mathbf{x})\mathbf{E}_p^M(\mathbf{x})\right).\end{aligned}\tag{3.17}$$

From the appendix of [5] it follows, on passing to a subsequence, if necessary, that for every $r>0$

$$\lim_{\varepsilon_k\to 0}\int_{Q(\mathbf{x},r)}|(-\nabla w_p^{k,r}(\mathbf{y}))-\mathbf{G}^p(T(\mathbf{y}/\varepsilon_k)\omega)|^2\,d\mathbf{y}=0 \tag{3.18}$$

for almost every ω. From this we deduce that for a denumerable sequence $\{r_j\}_{j=1}^{\infty}$, $r_j\to 0$

$$\begin{aligned}&\left(\sum_{m=1}^{3}\nabla_{mm}^{N+1}\hat{A}^E(A_1,A_2,\dots,A_N,A_i,\mathbf{x},\omega)\right)\mathbf{E}^M(\mathbf{x})\cdot\mathbf{E}^M(\mathbf{x})\\ &\quad=\lim_{r_j\to 0}\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y}/\varepsilon_k)\omega))\mathbf{E}^M(\mathbf{x})|^2\,d\mathbf{y}\right)\end{aligned}\tag{3.19}$$

for almost every ω. Applying the Hölder inequality gives

$$\begin{aligned}&\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y}/\varepsilon_k)\omega))\mathbf{E}^M(\mathbf{x})|^2\,d\mathbf{y}\\ &\quad\le\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)\,d\mathbf{y}\|\tilde{\chi}_i(T(\mathbf{y})\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y})\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^\infty(R^3)}\\ &\quad\le\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)\,d\mathbf{y}\|\tilde{\chi}_i(\omega)|(I+\underline{\tilde{\mathbf{G}}}(\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^\infty(\Omega)}.\end{aligned}\tag{3.20}$$

The last inequality in (3.20) follows from a straightforward argument given in the appendix. Noting that

$$\lim_{r_j\to 0}\lim_{\varepsilon_k\to 0}\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)\,d\mathbf{y}=\theta_{t,i}(\mathbf{x},\omega) \tag{3.21}$$

for almost all $\mathbf{x}$ and applying (3.20) to (3.19), we arrive at the estimate

$$\begin{aligned}&\left(\sum_{m=1}^{3}\nabla_{mm}^{N+1}\hat{A}^E(A_1,A_2,\dots,A_N,A_i,\mathbf{x},\omega)\right)\mathbf{E}^M(\mathbf{x})\cdot\mathbf{E}^M(\mathbf{x})\\&\quad\le\theta_{t,i}(\mathbf{x},\omega)f_\infty^i(\mathbf{E}^M(\mathbf{x}))\end{aligned} \tag{3.22}$$

for almost every ω in Ω, and the proof of (3.8) of Lemma 3.2 is complete. To prove (3.9) we return to (3.19) and apply the Hölder inequality with $1/p+1/q=1$ to obtain

$$\begin{aligned}&\lim_{\varepsilon_k\to 0}\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y}/\varepsilon_k)\omega))\mathbf{E}^M(\mathbf{x})|^2\,d\mathbf{y}\\&\quad\le\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\chi_{t,i}^{\varepsilon_k}(\mathbf{y},\omega)\,d\mathbf{y}\right)^{1/q}\\&\quad\quad\times\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\tilde{\chi}_i(T(\mathbf{y}/\varepsilon_k)\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y}/\varepsilon_k)\omega))\mathbf{E}^M(\mathbf{x})|^{2p}d\mathbf{x}\right)^{1/p}.\end{aligned} \tag{3.23}$$

From the Birkhoff ergodic theorem it follows that

$$\begin{aligned}&f_p^i(\mathbf{E}^M(\mathbf{x}))\\&\quad=\lim_{\varepsilon_k\to 0}\left(\frac{1}{|Q(\mathbf{x},r_j)|}\int_{Q(\mathbf{x},r_j)}\tilde{\chi}_i(T(\mathbf{y}/\varepsilon_k)\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y}/\varepsilon_k)\omega))\mathbf{E}^M(\mathbf{x})|^{2p}d\mathbf{x}\right)^{1/p},\end{aligned} \tag{3.24}$$

and the proof of (3.9) is complete. □

4. Homogenization of Chebyshev's inequality. In this section we establish Proposition 2.1. We start by providing the relationship between the limits $\lim_{\varepsilon_k\to 0}P_i^{\varepsilon_k}(t,C_0,\omega)$, $\lim_{\varepsilon_k\to 0}P^{\varepsilon_k}(t,C_0,\omega)$ and the distribution of states for the square of the electric field intensity in the ith phase. The volume of the subset of the ith phase contained in C_0 where the equivalent stress exceeds t is given by $\lambda_i^{\varepsilon_k}(t,C_0,\omega)=\int_{C_0}\chi_{t,i}^{\varepsilon_k}(\mathbf{x},\omega)d\mathbf{x}$. Passing to a subsequence if necessary, the theory of weak convergence delivers the distribution of states $\theta_{t,i}(\mathbf{x},\omega)$ for which $\lim_{\varepsilon_k\to 0}\lambda_i^{\varepsilon_k}(t,C_0,\omega)=\int_{C_0}\theta_{t,i}(\mathbf{x},\omega)d\mathbf{x}$. For fixed ε_k the volume of the ith phase in the cube C_0 is denoted by $V_i^{\varepsilon_k}$ and $P_i^{\varepsilon_k}(t,C_0,\omega)=\lambda_i^{\varepsilon_k}(t,C_0,\omega)/V_i^{\varepsilon_k}$. From ergodicity, $\lim_{\varepsilon_k\to 0}V_i^{\varepsilon_k}=\theta_i|C_0|$. It is clear that

$$\lim_{\varepsilon_k\to 0}P_i^{\varepsilon_k}(t,C_0,\omega)=\left(\frac{1}{\theta_i|C_0|}\right)\int_{C_0}\theta_{t,i}(\mathbf{x},\omega)\,d\mathbf{x}. \tag{4.1}$$

Set $\theta_t(\mathbf{x},\omega)=\sum_{i=1}^N\theta_{t,i}(\mathbf{x},\omega)$; then one has

$$\lim_{\varepsilon_k\to 0}P^{\varepsilon_k}(t,C_0,\omega)=(1/|C_0|)\int_{C_0}\theta_t(\mathbf{x},\omega)\,d\mathbf{x}. \tag{4.2}$$

It follows easily from the homogenization constraint (3.4) that

$$\theta_{t,i}(\mathbf{x},\omega) \le t^{-p}|f_p^i(\mathbf{E}^M(\mathbf{x}))|^p, \qquad i = 1,\dots,N. \tag{4.3}$$

Taking averages of both sides gives

$$\lim_{\varepsilon_k\to 0} P_i^{\varepsilon_k}(t, C_0, \omega) \le t^{-p}\left(\frac{1}{\theta_i|C_0|}\right)\int_{C_0} |f_p^i(\mathbf{E}^M(\mathbf{x}))|^p\, d\mathbf{x}, \tag{4.4}$$

and (2.10) of Proposition 2.1 is proved. The inequality (2.11) of Proposition 2.1 follows immediately upon summation of the left and right sides of (4.3) over $i = 1,\dots,N$ and averaging both sides.

5. Bounds on the support set of the electric field intensity distribution function. This section contains the proofs of Propositions 2.2 and 2.3. The homogenization constraint (3.3) is used to prove Proposition 2.2. Integration of (3.3) gives

$$\int_{C_0} \theta_{t,i}(\mathbf{x},\omega) f^i(\mathbf{E}^M(\mathbf{x}))\, d\mathbf{x} - t\int_{C_0} \theta_{t,i}(\mathbf{x},\omega)\, d\mathbf{x} \ge 0, \quad i = 1,\dots,N. \tag{5.1}$$

Application of Hölder's inequality to the first term and division by $\theta_i|C_0|$ gives

$$\lim_{\varepsilon_k\to 0} P_i^{\varepsilon_k}(t, C_0, \omega)(\|f^i(\mathbf{E}^M)\|_{L^\infty(C_0)} - t) \ge 0, \quad i = 1,\dots,N, \tag{5.2}$$

and Proposition 2.2 follows.

To prove Proposition 2.3 we add the constraints (5.1) to get

$$\sum_{i=1}^N \left(\int_{C_0} \theta_{t,i}(\mathbf{x},\omega) f^i(\mathbf{E}^M(\mathbf{x}))\, d\mathbf{x}\right) - t\int_{C_0} \theta_t(\mathbf{x},\omega)\, d\mathbf{x} \ge 0. \tag{5.3}$$

Noting that $M(\mathbf{E}^M(\mathbf{x})) \ge f^i(\mathbf{E}^M(\mathbf{x}))$ gives

$$\int_{C_0} \theta_t(\mathbf{x},\omega) M(\mathbf{E}^M(\mathbf{x}))\, d\mathbf{x} - t\int_{C_0} \theta_t(\mathbf{x},\omega)\, d\mathbf{x} \ge 0. \tag{5.4}$$

Application of Hölder's inequality to the first term and division by $|C_0|$ gives

$$\lim_{\varepsilon_k\to 0} P^{\varepsilon_k}(t, C_0, \omega)(\|M(\mathbf{E}^M(\mathbf{x}))\|_{L^\infty(C_0)} - t) \ge 0, \tag{5.5}$$

and Proposition 2.3 follows.

6. Pointwise bounds on the electric field intensity. In this section we give the proof of Proposition 2.4. From the hypothesis of Propositions 2.4 and 2.3 it follows that $\lim_{k\to\infty} |S_t^{\varepsilon_k}(\omega) \cap C_0| = 0$. We choose a subsequence $\{\varepsilon_{k'}\}_{k'=1}^\infty$ such that $|S_t^{\varepsilon_{k'}}(\omega) \cap C_0| < 2^{-k'}$. Then if $\mathbf{x}$ doesn't belong to $\cup_{k'\ge\tilde{K}}^\infty S_t^{\varepsilon_{k'}}(\omega) \cap C_0$, one has that $|\mathbf{E}^{\varepsilon_{k'}}|^2 \le t$ for every $k' > \tilde{K}$. Hence for any $\mathbf{x}$ not in $A = \cap_{K=1}^\infty \cup_{k'\ge K}^\infty S_t^{\varepsilon_{k'}}(\omega) \cap C_0$ there is an index K for which $|\mathbf{E}^{\varepsilon_{k'}}|^2 \le t$ for every $k' > K$. But

$$|A| \le \left|\cup_{k'\ge\tilde{K}}^\infty S_t^{\varepsilon_{k'}}(\omega) \cap C_0\right| \le \sum_{k'=\tilde{K}}^\infty |S_t^{\varepsilon_{k'}}(\omega) \cap C_0| \le 2^{-\tilde{K}+1}.$$

Hence $|A| = 0$. Thus for almost every $\mathbf{x}$ in C_0 there is a finite index K (that may depend upon $\mathbf{x}$ and ω) for which $|\mathbf{E}^{\varepsilon_{k'}}|^2 \le t$ for every $k' > K$, and the proposition follows.

7. Upper bounds on the stress distribution function. In this section Propositions 2.5 and 2.6 are derived. For a cube C_0 contained inside the composite, the set of points where $M(\mathbf{E}^M(\mathbf{x})) \geq t$ is denoted by $\{\mathbf{x} \text{ in } C_0;\ M(\mathbf{E}^M(\mathbf{x})) \geq t\}$. We start by establishing the inequality

$$\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \leq \frac{|\{\mathbf{x} \text{ in } C_0;\ M(\mathbf{E}^M(\mathbf{x})) \geq t\}|}{|C_0|}. \tag{7.1}$$

Adding the homogenization constraints gives

$$\theta_t(\mathbf{x}, \omega)(M(\mathbf{E}^M(\mathbf{x})) - t) \geq 0. \tag{7.2}$$

Thus from (7.2) it is evident that at almost every point for which $\theta_t(\mathbf{x}, \omega) > 0$, one has that $M(\mathbf{E}^M(\mathbf{x})) \geq t$. The set of points in C_0 for which $\theta_t(\mathbf{x}, \omega) > 0$ is denoted by $\{\mathbf{x} \text{ in } C_0;\ \theta_t(\mathbf{x}, \omega) > 0\}$, and it is clear that

$$|\{\mathbf{x} \text{ in } C_0;\ \theta_t(\mathbf{x}, \omega) > 0\}| \leq |\{\mathbf{x} \text{ in } C_0;\ M(\mathbf{E}^M(\mathbf{x})) \geq t\}|. \tag{7.3}$$

Since $0 \leq \theta_t(\mathbf{x}, \omega) \leq 1$, one has the estimate

$$\int_{C_0} \theta_t(\mathbf{x}, \omega)\, d\mathbf{x} \leq |\{\mathbf{x} \text{ in } C_0;\ \theta_t(\mathbf{x}, \omega) > 0\}|, \tag{7.4}$$

and (7.1) follows from (7.3).

We will apply the John–Nirenberg theorem [6] to estimate the right-hand side of (7.1). To do this we show first that

$$|\{\mathbf{x} \text{ in } C_0;\ M(\mathbf{E}^M(\mathbf{x})) \geq t\}| \leq |\{\mathbf{x} \text{ in } C_0;\ |M(\mathbf{E}^M(\mathbf{x})) - M_{C_0}| \geq t - M_{C_0}\}|. \tag{7.5}$$

To see this, note that $M(\mathbf{E}^M(\mathbf{x})) \leq |M(\mathbf{E}^M(\mathbf{x})) - M_{C_0}| + M_{C_0}$, so

$$\{\mathbf{x} \text{ in } C_0;\ M(\mathbf{E}^M(\mathbf{x})) \geq t\} \subset \{\mathbf{x} \text{ in } C_0;\ |M(\mathbf{E}^M(\mathbf{x})) - M_{C_0}| \geq t - M_{C_0}\}, \tag{7.6}$$

and (7.5) follows. Application of the John–Nirenberg theorem gives

$$\frac{|\{\mathbf{x} \text{ in } C_0;\ |M(\mathbf{E}^M(\mathbf{x})) - M_{C_0}| \geq s\}|}{|C_0|} \leq \begin{cases} 1 & \text{for } 0 < s \leq 8\|M\|_{BMO}\alpha^{-1}, \\ \alpha^{-1} e^{[-(C(\alpha)\times(s))]} & \text{for } 8\|M\|_{BMO}\alpha^{-1} < s. \end{cases} \tag{7.7}$$

Proposition 2.5 follows immediately from the change of variables $s = t - M_{C_0}$ and the inequalities (7.1), (7.5), and (7.7). The function obtained by the change of variables $s = t - M_{C_0}$ in (7.7) is denoted by $\overline{P}_\alpha(t, C_0)$, and

$$\overline{P}_\alpha(t, C_0) = \begin{cases} 1 & \text{for } 0 < t - M_{C_0} \leq 8\|M\|_{BMO}\alpha^{-1}, \\ \alpha^{-1} e^{[-(C(\alpha)\times(t - M_{C_0}))]} & \text{for } 8\|M\|_{BMO}\alpha^{-1} < t - M_{C_0}. \end{cases} \tag{7.8}$$

It is evident from the estimates that $\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \leq \overline{P}_\alpha(t, C_0)$ for $M_{C_0} < t$. Tighter upper bounds are given by optimizing over α, i.e.,

$$\lim_{\varepsilon_k \to 0} P^{\varepsilon_k}(t, C_0, \omega) \leq \overline{U}(t, C_0) = \inf_{0<\alpha<1} \overline{P}_\alpha(t, C_0). \tag{7.9}$$

Here $\overline{U}(t, C_0)$ is continuous and decreasing and is given by

(7.10)
$$\overline{U}(t, C_0) = \begin{cases} 1 & \text{for } 0 < t - M_{C_0} \leq 8\|M\|_{BMO}, \\ (\alpha(t))^{-1} e \times e^{[-\alpha(t)(t - M_{C_0})/(8\|M\|_{BMO})]} & \text{for } 8\|M\|_{BMO} + M_{C_0} < t. \end{cases}$$

The factor $\alpha(t)$ lies in the interval $e^{-1} < \alpha(t) < 1$ and is the root of the equation

$$\kappa^{-1} - \alpha(1 + \ln \alpha) = 0, \tag{7.11}$$

where $\kappa = (t - M_{C_0})/(8\|M\|_{BMO})$. Proposition 2.6 now follows immediately from (7.10).

8. Macrofield modulation functions for random two-phase layered composites. In this section we treat randomly layered media and give an example of how the macrofield modulation functions are used to assess the field distribution inside a finite size sample. We start by considering a two-dimensional electrostatic problem on the plane R^2 and derive explicit formulas for the moments of the electric field. The plane is partitioned into layers of unit thickness parallel to the y_2 axis. Each layer contains an isotropic dielectric material having either dielectric constant α or β with $\alpha < \beta$. The particular value of the dielectric constant in each layer is given by a Bernoulli process; i.e., a biased coin that takes heads with probability θ and tails with probability $1 - \theta$ is used to assign the dielectric constant in each layer. Over each layer the coin is flipped, and if the coin lands heads up, the layer is assigned the β dielectric; otherwise it assigned the α dielectric. In section 8.1 we calculate the moments of the electric field directly using the strong law of large numbers. In section 8.2 we apply these results and use Proposition 2.2 to assess the distribution of the electric field intensity inside an L-shaped domain filled with a highly oscillatory random laminate in the presence of a prescribed electric charge density.

8.1. Moments of the electric field for random two-phase layered composites. For a given infinite sequence of biased coin flips, we arrive at a realization of the random medium. The indicator function ω of the β phase is a function of the y_1 coordinate and takes the value one in the β phase and zero outside it. For convenience we choose the origin of the $y_1 - y_2$ coordinate system to lie on a two-phase interface, with the β phase on the left and the α phase on the right. The coordinates of the interfaces between α and β phases on the positive y_1 axis are given by the sequence $\{N_n\}_{n=1}^{\infty}$ and $N_0 = 0$. The coordinates of the interfaces between phases on the negative y_1 axis are given by $\{N_n\}_{n=-1}^{-\infty}$. Let $\mathbf{e}^1$ and $\mathbf{e}^2$ be unit vectors pointing in the directions of the y_1 axis and y_2 axis, respectively. For imposed electric field gradients $\mathbf{e}^k$, $k = 1, 2$, the fluctuating part of the electric potential φ^k is continuous and solves the two-dimensional version of the field problem (2.2) given by

(8.1)
$$\begin{aligned} &\Delta\varphi^k = 0 \quad \text{inside each layer}, \\ &\beta(\partial_{y_1}\varphi^k_{|_L} + \mathbf{e}^k_1) = \alpha(\partial_{y_1}\varphi^k_{|_R} + \mathbf{e}^k_1) \text{ on interfaces}. \end{aligned}$$

It is clear from the above that $\varphi^1 = \varphi^1(y_1)$ and $\varphi^2 = \text{const}$. In this context the analogue of (2.1) is given by

$$\lim_{r\to\infty} \frac{\int_{-r}^{r} \partial_{y_1}\varphi^1 dy_1}{2r} = \lim_{r\to\infty} \frac{\varphi^1(r) - \varphi^1(-r)}{2r} = 0 \tag{8.2}$$

and $\varphi^k(0) = 0$. Clearly $\varphi^2 = 0$, and the potential φ^1 is a continuous piecewise linear function of y_1, i.e., in each phase φ^1 is of the form $\varphi^1(y_1) = ay_1 + b$ where the constants a and b change between phases. Application of (8.1), the continuity conditions at two-phase interfaces, and (8.2) together with the strong law of large numbers shows that $\varphi^1(y_1)$ is given a.s. by the following formulas.

For $N_n \le y_1 < N_{n+1}$ and $n+1$ even, the potential is given by

$$\varphi^1(y_1) = (k_2 - 1)y_1 + k_1(N_1 - N_2 + N_3 - N_4 + \cdots + N_n), \tag{8.3}$$

and for $n+1$ odd

$$\varphi^1(y_1) = (k_3 - 1)y_1 + k_1(N_1 - N_2 + N_3 - N_4 + \cdots - N_n). \tag{8.4}$$

For $N_{-(n+1)} < y_1 \le N_{-n}$ and $n+1$ even, the potential is given by

$$\varphi^1(y_1) = (k_3 - 1)y_1 + k_1(-N_{-1} + N_{-2} - N_{-3} + N_{-4} + \cdots - N_{-n}), \tag{8.5}$$

and for $n+1$ odd

$$\varphi^1(y_1) = (k_2 - 1)y_1 + k_1(-N_{-1} + N_{-2} - N_{-3} + N_{-4} + \cdots + N_{-n}), \tag{8.6}$$

where the constants k_1, k_2, and k_3 are defined by

$$\begin{aligned} k_1 &= \frac{\beta - \alpha}{\alpha + (\beta - \alpha)(1 - \theta)}, \\ k_2 &= \frac{\alpha}{\alpha + (\beta - \alpha)(1 - \theta)}, \\ k_3 &= \frac{\beta}{\alpha + (\beta - \alpha)(1 - \theta)}. \end{aligned} \tag{8.7}$$

The derivative $\partial_{y_1}\varphi^1$ is given by the following formula:

$$\begin{aligned} \partial_{y_1}\varphi^1 &= \gamma_\alpha = \frac{\theta(\beta - \alpha)}{\alpha + (\beta - \alpha)(1 - \theta)} \text{ in the } \alpha \text{ phase}, \\ \partial_{y_1}\varphi^1 &= \gamma_\beta = \frac{-(1 - \theta)(\beta - \alpha)}{\alpha + (\beta - \alpha)(1 - \theta)} \text{ in the } \beta \text{ phase}. \end{aligned} \tag{8.8}$$

For an imposed constant applied field of the general form $\overline{\mathbf{E}} = E_1\mathbf{e}^1 + E_2\mathbf{e}^2$, the local electric field $\mathbf{E}(\mathbf{y}, \omega)$ is given by

$$\mathbf{E}(\mathbf{y}, \omega) = (1 - \omega(y_1))((1 + \gamma_\alpha)E_1\mathbf{e}^1 + E_2\mathbf{e}^2) + \omega(y_1)((1 + \gamma_\beta)E_1\mathbf{e}^1 + E_2\mathbf{e}^2). \tag{8.9}$$

We average over the plane and apply the strong law of large numbers to obtain the moments of the local electric field given by

$$\begin{aligned} f_p^1(\overline{\mathbf{E}}) &= \lim_{r\to\infty}\left(\frac{1}{2r}\int_{-r}^{r}(1 - \omega(y_1))|\mathbf{E}(\mathbf{y}, \omega)|^{2p}\,dy_1\right)^{1/p} \\ &= (1 - \theta)^{1/p}((1 + \gamma_\alpha)^2E_1^2 + E_2^2), \end{aligned} \tag{8.10}$$

$$\begin{aligned} f_p^2(\overline{\mathbf{E}}) &= \lim_{r\to\infty}\left(\frac{1}{2r}\int_{-r}^{r}\omega(y_1)|\mathbf{E}(\mathbf{y}, \omega)|^{2p}\,dy_1\right)^{1/p} \\ &= \theta^{1/p}((1 + \gamma_\beta)^2E_1^2 + E_2^2). \end{aligned} \tag{8.11}$$

FIG. 8.1. *A realization for* $\theta = 1/3$.

The local dielectric constant in the random laminate is given by

$$A(\mathbf{y},\omega) = \alpha(1-\omega(y_1)) + \beta\omega(y_1), \tag{8.12}$$

and the effective tensor A^E is given by

$$\begin{aligned} A^E\overline{\mathbf{E}} &= \lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r} A(\mathbf{y},\omega)\mathbf{E}(\mathbf{y},\omega)\,dy_1 \\ &= (\alpha^{-1}(1-\theta)+\beta^{-1}\theta)^{-1}E_1 + (\alpha(1-\theta)+\beta\theta)E_2. \end{aligned} \tag{8.13}$$

The random laminate described above is an example of a symmetric cell material [15]. A standard construction delivers the probability space $(\Omega,\mathcal{F},\mathcal{P})$ and dynamical system associated with the symmetric cell material; see [8, 17]. Using this, one rewrites the averages given in (8.10), (8.11), and (8.13) in terms of the ensemble averages used to define the moments of the local electric field and effective dielectric constant in section 2.

8.2. Electric field assessment for a randomly layered dielectric in an *L*-shaped domain. In this subsection we apply the theory presented in section 2 to assess the electric field distribution inside an L-shaped domain containing a highly oscillatory random laminate with length scale $\varepsilon_k = 1/k$, $k = 1, 2, \dots$. Here the L-shaped domain is taken to have side length one. The dielectric constant for the highly oscillatory random laminate inside the L-shaped domain is given by

$$A^{\varepsilon_k}(\mathbf{x},\omega) = A(x_1/\varepsilon_k,\omega), \tag{8.14}$$

where $A(\mathbf{y},\omega)$ is given by the Bernoulli process (8.12) with $\theta = 1/3$. A realization of the random laminate with characteristic length scale ε_{40} is given in Figure 8.1. Here the subdomain in white is the α dielectric and the subdomain in black is the β dielectric.

The electric potential $\phi^{\varepsilon_k}(\mathbf{x},\omega)$ is the solution of

$$-\mathrm{div}(A^{\varepsilon_k}(\mathbf{x},\omega)\nabla\phi^{\varepsilon_k}(\mathbf{x},\omega)) = 10 \tag{8.15}$$

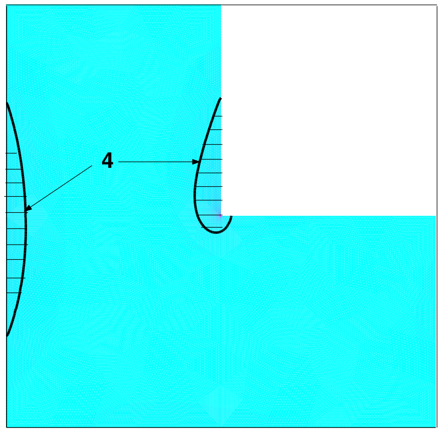

FIG. 8.2. *Distribution of the electric field intensity in the α dielectric.*

inside the L-shaped domain and $\phi^{\varepsilon_k}(\mathbf{x},\omega) = 0$ on the boundary. The associated electric field is given by $\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega) = -\nabla\phi^{\varepsilon_k}(\mathbf{x},\omega)$. The goal of this application is to characterize the distributions $\lim_{\varepsilon_k\to 0} P_i^{\varepsilon_k}(t, C_0, \omega)$, $i = 1, 2$. Here C_0 can be any square contained inside the L-shaped domain. To do this we solve numerically for the macroscopic potential and electric field and construct the macrofield modulation functions. The macroscopic electric potential $\phi^M(\mathbf{x})$ satisfies the boundary condition $\phi^M(\mathbf{x}) = 0$ and

$$-\text{div}(A^E\nabla\phi^M(\mathbf{x})) = 10. \tag{8.16}$$

The macroscopic electric field is given by $\mathbf{E}^M(\mathbf{x}) = -\nabla\phi^M(\mathbf{x})$. The macrofield modulation functions are given by

$$f_p^1(\mathbf{E}^M(\mathbf{x})) = (1-\theta)^{1/p}((1+\gamma_\alpha)^2|\partial_{x_1}\phi^M(\mathbf{x})|^2 + |\partial_{x_2}\phi^M(\mathbf{x})|^2), \tag{8.17}$$

$$f_p^2(\mathbf{E}^M(\mathbf{x})) = \theta^{1/p}((1+\gamma_\beta)^2|\partial_{x_1}\phi^M(\mathbf{x})|^2 + |\partial_{x_2}\phi^M(\mathbf{x})|^2). \tag{8.18}$$

For the computation we choose $\alpha = 2$ and $\beta = 10$ and restrict our attention to $f_\infty^1(\mathbf{E}^M(\mathbf{x}))$ and $f_\infty^2(\mathbf{E}^M(\mathbf{x}))$. To illustrate the ideas, the level curves given by $f_\infty^1(\mathbf{E}^M(\mathbf{x})) = 4$ are plotted in Figure 8.2. The lined regions indicate where $f_\infty^1(\mathbf{E}^M(\mathbf{x})) > 4$ and $f_\infty^1(\mathbf{E}^M(\mathbf{x})) < 4$ outside these. For any square C_0 that doesn't intersect the lined regions, Proposition 2.2 implies that

$$\lim_{\varepsilon_k\to 0} P_1^{\varepsilon_k}(t, C_0, \omega) = 0 \quad \text{for } t > 4 \tag{8.19}$$

for almost every realization ω. In this way it is seen that the lined regions provide an asymptotically exact bound on the set where $|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2 > 4$ in the α dielectric.

The level curves given by $f_\infty^2(\mathbf{E}^M(\mathbf{x})) = 1$ are plotted in Figure 8.3. The lined regions indicate where $f_\infty^2(\mathbf{E}^M(\mathbf{x})) > 1$ and $f_\infty^1(\mathbf{E}^M(\mathbf{x})) < 1$ outside these. For any square C_0 that doesn't intersect the lined regions, Proposition 2.2 implies that

$$\lim_{\varepsilon_k\to 0} P_2^{\varepsilon_k}(t, C_0, \omega) = 0 \quad \text{for } t > 1 \tag{8.20}$$

for almost every realization ω. It is seen as before that the lined regions provide an asymptotically exact bound on the set where $|\mathbf{E}^{\varepsilon_k}(\mathbf{x},\omega)|^2 > 1$ in the β dielectric.

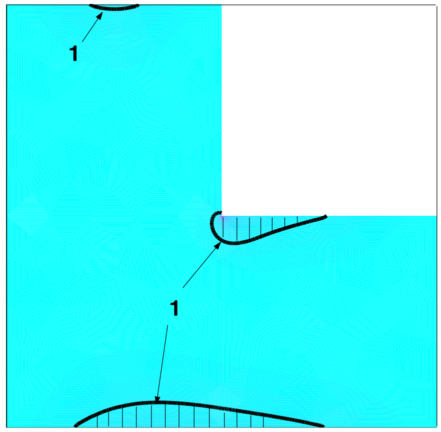

FIG. 8.3. *Distribution of the electric field intensity in the β dielectric.*

Appendix. Here we establish the inequality stated in (3.20) given by

$$\begin{aligned} &\|\tilde{\chi}_i(T(\mathbf{y})\omega)|(I+\underline{\tilde{\mathbf{G}}}(T(\mathbf{y})\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^\infty(R^3)} \\ &\quad \leq \|\tilde{\chi}_i(\omega)|(I+\underline{\tilde{\mathbf{G}}}(\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^\infty(\Omega)} \end{aligned} \tag{A.1}$$

for almost every ω. To establish (A.1) put $\alpha = \|\tilde{\chi}_i(\omega)|(I+\underline{\tilde{\mathbf{G}}}(\omega))\mathbf{E}^M(\mathbf{x})|^2\|_{L^\infty(\Omega)}$ and introduce the set $\mathcal{G} = \{\omega : \tilde{\chi}_i(\omega)|(I+\underline{\tilde{\mathbf{G}}}(\omega))\mathbf{E}^M(\mathbf{x})|^2 \leq \alpha\}$. From Lemma 7.1 of [8] there exists a set $\mathcal{G}_1 \subset \mathcal{G}$ for which $\mathcal{P}(\mathcal{G}_1) = 1$, and for any fixed ω in $\mathcal{G}_1$, one has that $T(\mathbf{y})\omega$ is in $\mathcal{G}$ for almost every $\mathbf{y}$ in R^3, and (A.1) follows.

REFERENCES

[1] A. BENSOUSSAN, J. L. LIONS, AND G. PAPANICOLAOU, *Asymptotic Analysis for Periodic Structures*, Stud. Math. Appl. 5, North-Holland, Amsterdam, 1978.
[2] A. BOURGREAT, A. MIKELIĆ, AND S. WRIGHT, *Stochastic two-scale convergence in the mean and applications*, J. Reine Angew. Math., 456 (1994), pp. 19–51.
[3] L. BREIMAN, *Probability*, Addison-Wesley, Reading, MA, 1968.
[4] H. CHENG AND S. TORQUATO, *Electric field fluctuations in random dielectric composites*, Phys. Rev. B, 56 (1997), pp. 8060–8068.
[5] K. GOLDEN AND G. PAPANICOLAOU, *Bounds for effective parameters of heterogeneous media by analytic continuation*, Comm. Math. Phys., 90 (1983), pp. 473–491.
[6] F. JOHN AND L. NIRENBERG, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., 14 (1961), pp. 415–426.
[7] D. JEULIN, *Random structure models for composite media and fracture statistics*, in Advances in Mathematical Modeling of Composite Materials, K. Z. Markov, ed., Advances in Mathematics for Applied Sciences 15, World Scientific, Singapore, 1994, pp. 239–289.
[8] V. V. JIKOV, S. M. KOZLOV, AND O. A. OLEINIK, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, Berlin, Heidelberg, London, New York, 1994.
[9] A. KELLY AND N. H. MACMILLAN, *Strong Solids*, Monographs on the Physics and Chemistry of Materials, Clarendon Press, Oxford, 1986.
[10] S. M. KOZLOV, *Averaging of random structures*, Dokl. Akad. Nauk SSSR, 241 (1978), pp. 1016–1019 (in Russian); Soviet Math. Dokl., 19 (1978), pp. 950–954 (in English).
[11] R. LIPTON, *Assessment of the local stress state through macroscopic variables*, R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci., 361 (2003), pp. 921–946.
[12] R. LIPTON, *Relaxation through homogenization for optimal design problems with gradient constraints*, J. Optim. Theory Appl., 114 (2002), pp. 27–53.
[13] R. LIPTON, *Stress constrained G closure and relaxation of structural design problems*, Quart. Appl. Math., 62 (2004), pp. 295–321.

[14] R. Lipton, *Bounds on the distribution of extreme values for the stress in composite materials*, J. Mech. Phys. Solids, 52 (2004), pp. 1053–1069.

[15] G. W. Milton, *The Theory of Composites*, Cambridge University Press, Cambridge, UK, 2002.

[16] F. Murat and L. Tartar, *H convergence*, in Topics in the Mathematical Modelling of Composite Materials, A. V. Cherkaev and R. V. Kohn, eds., Birkhäuser, Boston, 1997, pp. 21–43.

[17] G. Papanicolaou and S. R. S. Varadhan, *Boundary value problems with rapidly oscillating random coefficients*, Colloquia Mathematica Sociietatis János Bolyai, 27 (1982), pp. 835–873.

[18] E. Sanchez-Palencia, *Non-Homogeneous Media and Vibration Theory*, Springer-Verlag, Berlin, 1980.

[19] S. Spagnolo, *Convergence in energy for elliptic operators*, in Proceedings of the Third Symposium on Numerical Solutions of Partial Differential Equations, B. Hubbard, ed., Academic Press, New York, 1976, pp. 469–498.